

A rigidity theorem for *Moor*-bialgebras¹

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Abstract: We introduce the operad *Moor*, dual of the operad *NAP* and the notion of *Moor*-bialgebras. We warn the reader that the compatibility relation linking the *Moor*-operation with the *Moor*-cooperation is not distributive in the sense of Loday. Nevertheless, a rigidity theorem (à la Hopf-Borel) for the category of connected *Moor*-bialgebras is given. We show also that free permutative algebras can be equipped with a *Moor*-cooperation whose compatibility with the permutative product looks like the infinitesimal relation.

Notation: In the sequel K is a characteristic zero field and Σ_n is the group of permutations over n elements. If \mathcal{A} is an operad, then the K -vector space of n -ary operations is denoted as usual by $\mathcal{A}(n)$. We adopt Sweedler notation for the binary cooperation Δ on a K -vector space V and set $\Delta(x) = x_{(1)} \otimes x_{(2)}$.

1 Introduction

The well-known Hopf-Borel theorem states that any connected cocommutative commutative bialgebra (Hopf algebra) \mathcal{H} is free and cofree over its primitive part $\text{Prim } \mathcal{H}$. Otherwise stated;

Theorem 1.1 (*Hopf-Borel*) *For any cocommutative commutative bialgebra \mathcal{H} the following is equivalent.*

1. \mathcal{H} is connected;
2. \mathcal{H} is isomorphic to $\text{Com}(\text{Prim } \mathcal{H})$ as a bialgebra;
3. \mathcal{H} is isomorphic to $\text{Com}^c(\text{Prim } \mathcal{H})$ as a coalgebra.

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In the theory developed by J.-L. Loday, this result is rephrased by saying that the triple of operads $(Com, Com, Vect)$ is good. Some good triples of operads of type $(\mathcal{A}, \mathcal{A}, Vect)$ or $(\mathcal{C}, \mathcal{A}, Vect)$ have been found since and a summary can be found in [6]. The aim of this paper is to produce another good triple of operads of this form but without using the powerful theorems of J.-L. Loday [6], simply because his first Hypothesis $(H0)$ is not fulfilled by our objects.

From our coalgebra framework on weighted directed graphs [4, 3], we describe a directed graph by two cooperations Δ_M and $\tilde{\Delta}_M$ verifying:

$$(\tilde{\Delta}_M \otimes id)\Delta_M = (id \otimes \Delta_M)\tilde{\Delta}_M.$$

To code a bidirected graph, we have to add the extra condition $\tau\Delta_M = \tilde{\Delta}_M$, where τ is the usual flip map. The previous equation becomes,

$$(id \otimes \tau)(\Delta_M \otimes id)\Delta_M = (\Delta_M \otimes id)\Delta_M.$$

Such coalgebras were called L -cocommutative in [4]. On the algebra side, this yield K -vector spaces equipped with a binary operation \prec verifying,

$$(x \prec y) \prec z = (x \prec z) \prec y.$$

Such algebras came out in the work of M. Livernet [5] under the name nonassociative permutative algebras, NAP -algebras for short. The operad NAP of NAP -algebras is important because it is related to the operad $preLie$ of preLie-algebras. Indeed, the triple of operads $(NAP, preLie, Vect)$ has been shown to be good by M. Livernet [5]. Requiring the operation \prec to be associative leads to permutative algebras, or $Perm$ -algebras for short [1]. In this paper, we introduce the dual, in the sense of Ginzburg and Kapranov [6], of NAP -algebras, called $Moor$ -algebras in Sections 2-3 and give a rigidity theorem for the category of connected $Moor$ -bialgebras in Section 4, that is the triple of operads $(Moor, Moor, Vect)$ is good. This category is interesting, as we said, for we cannot apply the powerful results of J.-L. Loday [6] since the compatibility relation linking the cooperation and the operation of a $Moor$ -bialgebra is not distributive as required in [6], Hypothesis $(H0)$. We end with Section 5, where we show that the free permutative algebra over a K -vector space V can be equipped with a $Moor$ -cooperation whose compatibility relation with the permutative product looks like the nonunital infinitesimal relation.

2 On $Moor$ -algebras

Define the operad $Moor$, ($Moor$ because a typical element of a $Moor$ -algebra looks like $(\dots((x_1x_2)x_3)\dots)x_n$ whose parentheses are concentrating at the beginning, reminding boats being moored one behind the other) to be the free operad on one binary operation \prec divided out by the following set of relations:

$$R := \{(x \prec y) \prec z - (x \prec z) \prec y; \ x \prec (y \prec z)\}.$$

If V stands for a K -vector space, then $S(V)$ stands for the symmetric module over V , that is:

$$S(V) := K \oplus \bigoplus_{n>0} S^n(V),$$

where $S^n(V)$ is the quotient of $V^{\otimes n}$ by the usual action of the symmetric group Σ_n . A typical element of $S^n(V)$ will be written $v_1 \vee v_2 \vee \dots \vee v_n$, where the $v_i \in V$.

Theorem 2.1 *The following hold.*

1. *The dual of the operad NAP is the operad $Moor$.*
2. *The free $Moor$ -algebra over a K -vector space V is $V \otimes S(V)$ as a K -vector space equipped with the operation \prec defined by:*

$$v \otimes \omega \prec v' \otimes \omega' = v \otimes \omega \vee v',$$

if $\omega' \in K$ and vanishes otherwise.

3. *The generating series of the operad $Moor$ is,*

$$f_{Moor}(x) := xe^x = \sum_{n>0} n \frac{x^n}{n!}.$$

Proof: Observe that the free binary operad \mathcal{F} with one binary operation obey the relation $\dim \mathcal{F}(3) = 12$. We get $\dim NAP(3) = 9$ and $\dim Moor(3) = 3$. As in $\mathcal{F}(3)$, quadratic relations defining $NAP(3)$ are orthogonal (see [2]) to those defining $Moor(3)$, the dual of NAP is $Moor$. Let V be a K -vector space. The K -vector space $V \otimes S(V)$ equipped with the operation,

$$\prec: V \otimes S(V) \bigotimes V \otimes S(V) \rightarrow V \otimes S(V), \quad v \otimes \omega \prec v' \otimes \omega' = v \otimes \omega \vee v',$$

if $\omega' \in K$ and vanishes otherwise is a $Moor$ -algebra. Observe that $i: V \hookrightarrow V \otimes K \hookrightarrow V \otimes S(V)$ defined by $i(v) := v \otimes 1_K$ realises the expected embedding. Let (A, \prec_A) be a $Moor$ -algebra and $f: V \rightarrow A$ be a map. Define $\tilde{f}: V \otimes S(V) \rightarrow A$ by,

$$\tilde{f}(v \otimes 1_K) := f(v),$$

$$\tilde{f}(v \otimes v_1 \cdots v_p) := (\cdots ((f(v) \prec_A f(v_1)) \prec_A f(v_2)) \cdots \prec_A f(v_{p-1})) \prec_A f(v_p).$$

Then, \tilde{f} is a $Moor$ -morphism and the only one such that $\tilde{f} \circ i = f$. For the last item, observe that in a $Moor$ -algebra only these monomials,

$$(left\ combs : (lc)) \quad (\cdots (v_1 \prec v_2) \prec v_3) \cdots \prec v_{n-1} \prec v_n),$$

do not vanish. Indeed, one can model n -ary operations of the $Moor$ -operad with planar rooted binary trees whose nodes are decorated by \prec . For instance, $x \prec (y \prec z)$ is represented by $\searrow \swarrow$ so $\searrow \swarrow = 0$ and only left combs survive. Therefore, we get $n(n-1)!$ such left combs but because of the relation $(x \prec y) \prec z = (x \prec z) \prec y$, the relation (lc) is invariant under the action of the symmetric group Σ_{n-1} . Hence, $\dim Moor(n) = n$. \square

3 The cofree *Moor*-coalgebra

Let i be an integer. By v^i , we mean $v \vee \dots \vee v$, times i . In the sequel, we set by induction, for all $n > 0$, $\Delta_{\mathcal{H}}^{(1)} = \Delta_{\mathcal{H}}$ and $\Delta_{\mathcal{H}}^{(n)} := (\Delta_{\mathcal{H}} \otimes id_{(n-1)})\Delta_{\mathcal{H}}^{(n-1)}$ for any cooperation $\Delta_{\mathcal{H}}$ of a *Moor*-coalgebra \mathcal{H} . We get the following two propositions by dualising the corresponding results in the proof of Theorem 2.1.

Lemma 3.1 *Let $(\mathcal{H}, \Delta_{\mathcal{H}})$ be a coalgebra whose cooperation verifies $\Delta_{\mathcal{H}}^{(2)} = (id \otimes \tau)\Delta_{\mathcal{H}}^{(2)}$. For all $n > 0$, the map:*

$$\Delta_{\mathcal{H}}^{(n)} : \mathcal{H} \rightarrow \mathcal{H}^{\otimes(n+1)}, \quad x \mapsto \Delta_{\mathcal{H}}^{(n)}(x) := x_{n+1} \otimes x_n \otimes \dots \otimes x_i \otimes \dots \otimes x_2 \otimes x_1,$$

has its last n components invariant by Σ_n .

Proof: Fix $i = 1, \dots, n-1$. The following,

$$\begin{aligned} \Delta_{\mathcal{H}}^{(n)} &= (\Delta_{\mathcal{H}}^{(n-i-1)} \otimes id_{(i+1)}) \circ (\Delta_{\mathcal{H}}^{(2)} \otimes id_{(i-1)}) \circ \Delta_{\mathcal{H}}^{(i-1)}, \\ &= (\Delta_{\mathcal{H}}^{(n-i-1)} \otimes id_{(i+1)}) \circ ((id \otimes \tau)\Delta_{\mathcal{H}}^{(2)} \otimes id_{(i-1)}) \circ \Delta_{\mathcal{H}}^{(i-1)}, \\ &= (id_{(n-i)} \otimes \tau \otimes id_{(i-1)}) \circ (\Delta_{\mathcal{H}}^{(n-i-1)} \otimes id_{(i+1)}) \circ (\Delta_{\mathcal{H}}^{(2)} \otimes id_{(i-1)}) \circ \Delta_{\mathcal{H}}^{(i-1)}, \end{aligned}$$

shows that the last n components of $\Delta_{\mathcal{H}}^{(n)}$ are invariant by the transpositions $(i, i+1)$ for all $i = 1, \dots, n-1$, hence the claim. \square

Proposition 3.2 *The cofree *Moor*-coalgebra over a K -vector space V is:*

$$Moor^c(V) := V \otimes S(V),$$

as a K -vector space equipped with the following co-operation δ defined as follows:

$$\begin{aligned} \delta(v \otimes 1_K) &= 0, \\ \delta(v_1 \otimes v_2^{i_2} \vee \dots \vee v_n^{i_n}) &= \sum_{k=2}^n (v_1 \otimes v_2^{i_2} \vee \dots \vee v_k^{i_k-1} \vee \dots \vee v_n^{i_n}) \otimes (v_k \otimes 1_K). \end{aligned}$$

Let $\Gamma V^{\otimes n}$ be the K -vector space of tensors invariant through the usual action of Σ_n . For all $n > 1$, define $j_n : V \otimes \Gamma V^{\otimes n} \rightarrow V \otimes S^n(V)$ by $j_n(\sum_{\sigma \in \Sigma_n} v \otimes v_1 \otimes \dots \otimes v_n) = v \otimes v_1 \vee \dots \vee v_n$. They are bijective maps since K is a characteristic zero field.

Proposition 3.3 *If $(\mathcal{H}, \Delta_{\mathcal{H}})$ is a *Moor*^c-coalgebra and $f : \mathcal{H} \rightarrow V$ a linear map, set by induction $f^{\otimes 1} = f$ and $f^{\otimes n} = f^{\otimes(n-1)} \otimes f$. Then, the map $\tilde{f} : \mathcal{H} \rightarrow Moor^c(V)$ defined by:*

$$\tilde{f} := \sum_{n=1}^{\infty} j_n \circ f^{\otimes(n+1)} \circ \Delta_{\mathcal{H}}^{(n)},$$

is the unique coalgebra morphism verifying $\pi \circ \tilde{f} = f$, where $\pi : Moor^c(V) \twoheadrightarrow V$ is the canonical projection.

4 On *Moor*-bialgebras

4.1 Definition

In the sequel, we set for any $v_1, \dots, v_n \in V$:

$$(\dots (v_1 \prec v_2) \prec \dots) \prec v_n := [v_1 | v_2, \dots, v_n].$$

By definition, a *Moor*-bialgebra \mathcal{H} is the data of:

1. A graduated *Moor*-algebra $\mathcal{H} := \bigoplus_{p \geq 0} \mathcal{H}_p$,
2. A *Moor*-cooperation $\Delta_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{H}^{\otimes 2}$, i.e., verifying:

$$(id \otimes \Delta_{\mathcal{H}})\Delta_{\mathcal{H}} = 0,$$

$$(\Delta_{\mathcal{H}} \otimes id)\Delta_{\mathcal{H}} = (id \otimes \tau)(\Delta_{\mathcal{H}} \otimes id)\Delta_{\mathcal{H}},$$

3. The *Moor*-operation and cooperation have to be related by the following compatibility condition:

$$\Delta_{\mathcal{H}}(x \prec y) = x \otimes e(y) + (x_{(1)} \prec y) \otimes x_{(2)},$$

for any $x, y \in \mathcal{H}$, where $e : \mathcal{H} \rightarrow \mathcal{H}_1$ is the canonical projection.

A morphism of *Moor*-bialgebras is a morphism of graduated *Moor*-algebras and a morphism of *Moor*-coalgebras. Observe that this compatibility relation is not distributive in the sense of J.-L. Loday [6]. By *Prim* $\mathcal{H} := \ker \Delta_{\mathcal{H}}$ we mean the K -vector space of primitive elements.

Proposition 4.1 *Let \mathcal{H} be a *Moor*-bialgebra. Then, $\ker \Delta_{\mathcal{H}} = \tilde{\mathcal{H}}_1 \oplus \tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}} := \bigoplus_{j \in J} \tilde{\mathcal{H}}_j$, with J a suitable subset of $\mathbb{N} \setminus \{0, 1\}$. If it exists, $\tilde{\mathcal{H}}$ is a *Moor*-algebra equipped with the following action:*

$$\tilde{\mathcal{H}}_1 \otimes \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}, \quad h_1 \otimes h \mapsto h_1 \prec h.$$

Moreover, $\mathcal{H}_* := \bigoplus_{p \geq 1} \mathcal{H}_p$ acts on $\ker \Delta_{\mathcal{H}}$ on the right via:

$$\ker \Delta_{\mathcal{H}} \otimes \mathcal{H}_* \rightarrow \ker \Delta_{\mathcal{H}}, \quad a \otimes h \mapsto a \prec h.$$

Proof: For $j > 0$, set $\tilde{\mathcal{H}}_j$ the K -vector space of primitive elements of degree j . For h_j and $h_{j'}$ two primitive elements of degrees resp. $j \geq 1$ and $j' > 1$, one has:

$$\Delta_{\mathcal{H}}(h_j \prec h_{j'}) = h_j \otimes e(h_{j'}) = 0,$$

hence $h_j \prec h_{j'}$ of degree $j + j'$ is primitive. If $h \in \mathcal{H}_*$, then $\Delta(h_j \prec h) = h_j \otimes e(h) = 0$ hence the last claim. \square

Set $\Delta_{\mathcal{H}}^{(n)} := (\Delta_{\mathcal{H}} \otimes id_{(n-1)})\Delta_{\mathcal{H}}^{(n-1)}$ with $id_{(n-1)} = \underbrace{id \otimes \dots \otimes id}_{\text{times } n-1}$ for all $n \geq 1$. By definition, a *Moor*-bialgebra is said to be connected if $\mathcal{H} = \cup_{r>0} \mathcal{F}_r \mathcal{H}$, where the filtration $(\mathcal{F}_r \mathcal{H})_{r>0}$ is defined as follows:

$$(The \text{ primitive elements :}) \text{ Prim } \mathcal{H} := \mathcal{F}_1 \mathcal{H} := \ker \Delta_{\mathcal{H}} \subset \mathcal{H}_1.$$

Set $\Delta_{\mathcal{H}}^{(n)} := (\Delta_{\mathcal{H}} \otimes id_{(n-1)})\Delta_{\mathcal{H}}^{(n-1)}$ with $id_{(n-1)} = \underbrace{id \otimes \dots \otimes id}_{\text{times } n-1}$ for all $n \geq 1$. Then,

$$F_r \mathcal{H} := \ker \Delta_{\mathcal{H}}^{(r)}.$$

Here is an example of connected *Moor*-bialgebras.

Theorem 4.2 *Let V be a K -vector space. The free *Moor*-algebra over V is a connected *Moor*-bialgebra.*

Proof: Let V be a K -vector space. Define the co-operation Δ by induction as follows:

$$\Delta(v \otimes 1_K) := 0,$$

$$\Delta(x \prec y) = x \otimes \pi(y) + (x_{(1)} \prec y) \otimes x_{(2)},$$

for any $v \in V$, $x, y \in \text{Moor}(V)$, where $\pi : \text{Moor}(V) \rightarrow i(V)$ is the canonical projection map. As $x \prec y = 0$ for all $x, y \in \text{Moor}(V)$ and y of degree at least 2, we have to check that $\Delta(x \prec y)$ vanishes. But,

$$\Delta(x \prec y) = x \otimes \pi(y) + (x_{(1)} \prec y) \otimes x_{(2)} = 0,$$

because $\pi(y) = 0$ since the degree of y is at least 2 and $x_{(1)} \prec y = 0$ for the same reason. By induction one proves:

$$\Delta([v_1 | v_2, \dots, v_n]) = \sum_{k=2}^n [v_1 | v_2, \dots, \hat{v}_k, \dots, v_n] \otimes (v_k \otimes 1_K),$$

for all $v_1, \dots, v_n \in V$ and where the hat notation means as usual that the involved element vanishes. From that formula, it is straightforward to check that the co-operation Δ verifies:

$$\begin{aligned} (id \otimes \Delta)\Delta &= 0, \\ (\Delta \otimes id)\Delta &= (id \otimes \tau)(\Delta \otimes id)\Delta. \end{aligned}$$

Hence, the free *Moor*-algebra over V , which is graduated by construction, is a *Moor*-bialgebra.

The map $\phi(V) : Moor(V) \rightarrow Moor^c(V)$, defined as follows:

$$\phi(V)(v \otimes 1_K) = v \otimes 1_K,$$

$$\phi(V)([v_1|v_2^{i_2}, \dots, v_n^{i_n}]) = i_2! \dots i_n! v_1 \otimes v_2^{i_2} \vee \dots \vee v_n^{i_n},$$

is an isomorphism of *Moor*-coalgebras. It suffices to observe that:

$$\delta(v_1 \otimes v_2^{i_2} \vee \dots \vee v_n^{i_n}) = \sum_{k=2}^n (v_1 \otimes v_2^{i_2} \vee \dots \vee v_k^{i_k-1} \vee \dots \vee v_n^{i_n}) \otimes (v_k \otimes 1_K),$$

and:

$$\Delta([v_1|v_2^{i_2}, \dots, v_n^{i_n}]) = \sum_{k=2}^n i_k [v_1|v_2^{i_2} \vee \dots \vee v_k^{i_k-1} \vee \dots \vee v_n^{i_n}] \otimes (v_k \otimes 1_K),$$

Thus:

$$\delta(\phi(V)([v_1|v_2^{i_2}, \dots, v_n^{i_n}])) = i_2! \dots i_n! \sum_{k=2}^n (v_1 \otimes v_2^{i_2} \vee \dots \vee v_k^{i_k-1} \vee \dots \vee v_n^{i_n}) \otimes (v_k \otimes 1_K),$$

and:

$$\begin{aligned} & (\phi(V) \otimes \phi(V))\Delta([v_1|v_2^{i_2}, \dots, v_n^{i_n}]) = \\ &= (\phi(V) \otimes \phi(V))\left(\sum_{k=2}^n i_k [v_1|v_2^{i_2}, \dots, v_k^{i_k-1}, \dots, v_n^{i_n}] \otimes (v_k \otimes 1_K)\right) \\ &= \sum_{k=2}^n i_2! \dots i_k(i_k - 1)! \dots i_n! (v_1 \otimes v_2^{i_2} \vee \dots \vee v_k^{i_k-1} \vee \dots \vee v_n^{i_n}) \otimes (v_k \otimes 1_K). \end{aligned}$$

Hence, $\phi(V)$ is a coalgebra morphism and is bijective since K is a characteristic zero field. Therefore, $\ker \Delta = (Moor(V))_1$ and the filtration being given by the $((Moor(V))_n)_{n \geq 0}$, the free *Moor*-algebra over V is a connected *Moor*-bialgebra. \square

Lemma 4.3 *A connected Moor-bialgebra \mathcal{H} is generated by its primitive elements. Moreover, $\ker \Delta_{\mathcal{H}} = \mathcal{H}_1$.*

Proof: Let $x \in \mathcal{F}_r \mathcal{H}$ with r minimal and belongs to \mathcal{H}_p , $p > 0$ which is not primitive. Write $\Delta_{\mathcal{H}}(x) = x_{(1)} \otimes x_{(2)}$ as a sum of independents vectors. We get $0 = (id \otimes \Delta_{\mathcal{H}})\Delta_{\mathcal{H}}(x) = x_{(1)} \otimes \Delta_{\mathcal{H}}(x_{(2)})$. Hence $\Delta_{\mathcal{H}}(x_{(2)}) = 0$ and the $x_{(2)}$ are primitive elements and belongs to \mathcal{H}_1 . Moreover, $0 = \Delta_{\mathcal{H}}^{(r)}(x) = \Delta_{\mathcal{H}}^{(r-1)}(x_{(1)}) \otimes x_{(2)}$ which leads to $\Delta_{\mathcal{H}}^{(r-1)}(x_{(1)}) = 0$ and the $x_{(1)} \in \mathcal{F}_{r-1} \mathcal{H}$. Therefore,

$$\Delta_{\mathcal{H}}^{(r-1)}(x) = x_{(1)} \otimes \dots \otimes x_{(r)},$$

where the $x_{(i)}$ for $1 \leq i \leq r$ are primitive. However,

$$\Delta_{\mathcal{H}}^{(r-1)}(x - [x_{(1)}|x_{(2)}, \dots, x_{(r)}]) = 0,$$

hence $x - [x_{(1)}|x_{(2)}, \dots, x_{(r)}]$ is a primitive element and belongs to \mathcal{H}_1 . As $x \in \mathcal{H}_p$, $[x_{(1)}|x_{(2)}, \dots, x_{(r)}] \in \mathcal{H}_r$, we get $p = r$ and $x = [x_{(1)}|x_{(2)}, \dots, x_{(r)}]$. \square

4.2 A rigidity theorem for connected *Moor*-bialgebras

Theorem 4.4 *A connected Moor-bialgebra \mathcal{H} is free and cofree over its primitive part $\text{Prim } \mathcal{H}$, that is the following is equivalent for any Moor-bialgebra \mathcal{H} :*

1. \mathcal{H} is connected;
2. \mathcal{H} is isomorphic to $\text{Moor}(\text{Prim } \mathcal{H})$ as a Moor-bialgebra;
3. \mathcal{H} is isomorphic to $\text{Moor}^c(\text{Prim } \mathcal{H})$ as a Moor-coalgebra.

Proof: Let \mathcal{H} be a connected Moor-bialgebra. Since, $\text{Moor}(\text{Prim } \mathcal{H})$ is free, we get:

$$\begin{array}{ccc} \text{Prim } \mathcal{H} & \xrightarrow{i} & \text{Moor}(\text{Prim } \mathcal{H}) \\ & \searrow j & \downarrow \tilde{i} \\ & & \mathcal{H} \end{array}$$

where \tilde{i} is the unique Moor-morphism verifying $\tilde{i} \circ i = j$, where i and j are the canonical injections. Via Lemma 4.3, \tilde{i} is surjective. Since $\text{Moor}^c(\text{Prim } \mathcal{H})$ is cofree, we get,

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\tilde{e}} & \text{Moor}^c(\text{Prim } \mathcal{H}) \\ & \searrow e & \downarrow \pi \\ & & \text{Prim } \mathcal{H} \end{array}$$

with \tilde{e} the unique morphism of coalgebra extending the canonical projection e . Still set by induction $\Delta_{\mathcal{H}}^{(1)} = \Delta_{\mathcal{H}}$ and $\Delta_{\mathcal{H}}^{(n)} := (\Delta_{\mathcal{H}} \otimes \text{id}_{(n-1)}) \Delta_{\mathcal{H}}^{(n-1)}$. Set $e^{\otimes 1} = e$ and $e^{\otimes n} = e^{\otimes (n-1)} \otimes e$. Recall the coalgebraic morphism \tilde{e} is given as follows:

$$\tilde{e}(x) = \sum_{n=1}^{\infty} j_n \circ e^{\otimes (n+1)} \circ \Delta_{\mathcal{H}}^{(n)}(x).$$

As a connected Moor-bialgebra is generated by its primitive elements, we focus on elements $x := [x_1 | x_2, \dots, x_n]$, with the x_i primitive. As expected,

$$\tilde{e}([x_1 | x_2^{i_2}, \dots, x_n^{i_n}]_{\prec}) = i_2! \dots i_n! x_1 \otimes x_2^{i_2} \vee \dots \vee x_n^{i_n}.$$

Hence, we get on the whole \mathcal{H} , $\phi(\text{Prim } \mathcal{H}) = \tilde{e} \circ \tilde{i}$ where $\phi(\text{Prim } \mathcal{H})$ is defined in the proof of Theorem 4.2. Since \tilde{i} is surjective (Lemma 4.3) and $\phi(\text{Prim } \mathcal{H})$ is bijective, \tilde{i} is injective and is an isomorphism. Hence \tilde{e} is also an isomorphism of Moor-coalgebras since $\tilde{e} = \phi(\text{Prim } \mathcal{H}) \circ \tilde{i}^{-1}$. \square

5 A Moor-cooperation over free Perm-algebras

Permutative algebras have been introduced in [1]. In fact the following holds.

Proposition 5.1 *Let V be a K -vector space. Then, the K -vector space $V \otimes S(V)$ equipped with the operation \sqsupset defined by,*

$$v_1 \otimes v_2 \vee \dots \vee v_n \sqsupset w_1 \otimes w_2 \vee \dots \vee w_m = v_1 \otimes v_2 \vee \dots \vee v_n \vee w_1 \vee w_2 \vee \dots \vee w_m,$$

for all $v_i, w_j \in V$, is the free Perm-algebra over V .

A Perm-algebra P is said to be unital if it exists an element denoted by 1 such that $x \sqsupset 1 = x$, for all $x \in P$, the symbols $1 \sqsupset x$, $1 \sqsupset 1$ being not defined. The augmented free Perm-algebra over a K -vector space V , $K \oplus \text{Perm}(V)$, is a unital Perm-algebra.

Let V be a K -vector space. On $V \otimes S(V)$, one can define the left and right maps as follows:

$$l(1_K) = 0, \quad l(v_1 \otimes v_2 \vee \dots \vee v_n) = v_1 \otimes 1_K, \quad r(v_1 \otimes v_2 \vee \dots \vee v_n) = \frac{1}{n-1} \sum_{i=2}^n v_i \otimes v_2 \vee \dots \vee \hat{v}_i \vee \dots \vee v_n,$$

$$r(v_1 \otimes 1_K) = 1_K, \quad r(1_K) = 0,$$

for all $v_i \in V$.

Proposition 5.2 *Let V be a K -vector space. Then, the augmented free Perm-algebra over a K -vector space V , $K \oplus \text{Perm}(V)$, can be equipped with a Moor-cooperation Δ verifying the following compatibility relation:*

$$\Delta(1_K) = 0.$$

$$\Delta(x \sqsupset y) = (x_{(1)} \sqsupset y) \otimes x_{(2)} + (x \sqsupset y_{(1)}) \otimes y_{(2)} + (x \sqsupset r(y)) \otimes l(y),$$

for all $x, y \in K \oplus \text{Perm}(V)$.

Proof: Recall in $V \otimes S(V)$ the existence of the following Moor-cooperation Δ :

$$\Delta(v_1 \otimes v_2 \vee \dots \vee v_n) = \sum_{k=2}^n (v_1 \otimes v_2 \vee \dots \vee \hat{v}_k \vee \dots \vee v_n) \otimes (v_k \otimes 1_K),$$

defined for all $v_i \in V$. Add $\Delta(1_K) := 0$. Hence, set $x = v_1 \otimes v_2 \vee \dots \vee v_n$ and $y = w_1 \otimes w_2 \vee \dots \vee w_m$ and observe that

$$\Delta(x \sqsupset y) = (x_{(1)} \sqsupset y) \otimes x_{(2)} + (x \sqsupset y_{(1)}) \otimes y_{(2)} + (x \sqsupset r(y)) \otimes l(y),$$

holds. □

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